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Derivation of the Tully–Fisher Law: Doubts About the Necessity and Existence of Halo Dark Matter

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For disk galaxies the fourth power of the circular velocity v_c^4 of stars around the core of the galaxy is proportional to the luminosity L, $v_c^4 \propto L$ (Tully–Fisher law). Since L is proportional to the mass M of the galaxy, it follows that $v_c^4 \propto M$. Newtonian mechanics, however, yields $v_c^2 = GM/r$ for a circular motion. In order to rectify this big difference, astronomers assume the existence of dark matter. We derive the equation of motion of a star moving in the central field of a galaxy and show that, for a circular motion, it yields a term of the form $v_c^4 \propto GMc/\tau$, where G is Newton's gravitational constant, c is the speed of light, and τ is the Hubble time. This puts in doubt the existence of halo dark matter for galaxies.

1. INTRODUCTION

The universe is observed through electromagnetic waves. From the stars the waves are visible, from hot plasmas they are X-rays, from the hyperfine transition in hydrogen they are radio waves, and from the cosmic background radiation they are microwaves.

Not all matter in the universe, however, emits detectable radiation. Examples of this are black holes and zero-mass neutrinos. The difference between the detectable mass and the total mass that should be according to the laws of gravity is ascribed to the so-called dark matter, whose existence is inferred only from its gravitational interaction.

The visible parts of the galaxies are composed mainly of stars which do not satisfy Newton's mechanics and thus are hypothesized to be surrounded by extended halos of dark matter which may be a factor of 30 or more in both mass and size. The existence of the planet Neptune was predicted from the unexpected residuals in the motion of Uranus.

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A negative example, on the other hand, is the precession of the planet Murcury's perihelion. A hypothetical planet or a ring of matter inside Murcury's orbit was hypothesized to exist in order to explain the anomaly. No planet or material ring was observed. As is well known, the anomaly was resolved by Einstein's general relativity theory [1]. This is a reminder that much of the assumed missing matter might be explained by new theories.

In this paper we show that much of the unexplained observations can be satisfactorily described. More precisely, we prove that the Tully–Fisher law is included in the equations of motion obtained.

2. THE TULLY–FISHER LAW

Astronomical observations show that for disk galaxies the fourth power of the circular velocity of stars moving around the core of the galaxy, v_c^4, is proportional to the total luminosity L of the galaxy to an accuracy of more than two orders of magnitude in L, namely $v_c^4 \propto L$. Since L is proportional to the mass M of the galaxy, one obtains $v_c^4 \propto M$. This is known as the Tully–Fisher law [2]. There is no dependence on the distance of the star from the center of the galaxy as Newton's law $v_c^2 = GM/r$ requires for circular motion.

In order to rectify this deviation from Newton's laws, astronomers assume the existence of halos around the galaxy which are filled with dark matter and arranged in such a way as to satisfy the Tully–Fisher law for each particular situation.

It is well known that Newton's second law also follows from Einstein's general relativity theory in the lowest approximation in v/c, where v is a characteristic velocity and c the speed of light. For this reason we exclude the possibility of modifying Newton's second law of motion such as by adding to it a term which takes care of the anomaly [3, 4]. Any arbitrary modification of Newton's second law is therefore spurious even if it yields results that fit observations quite well.

3. THE HUBBLE LAW

The Hubble law asserts that faraway galaxies recede from each other at velocities proportional to their relative distances, $\mathbf{v} = H_0\mathbf{R}$, with $\mathbf{R} = (x, y, z)$. H_0 is a universal proportionality constant (at each cosmic time). Obviously the Hubble law can be written as ($\tau = H_0^{-1}$)

$$\tau^2 v^2 - (x^2 + y^2 + x^2) = 0 \tag{1}$$

and thus, when gravity is negligible, cosmology can be formulated as a special relativity with a new Lorentz-like transformation [5–9].

Gravitation, however, does not permit global linear relations like equation (1) and the latter has to be adapted to curved space. To this end one has to modify equation (1) to a differential form and to adjust it to curved space. The generalization of equation (1) is, accordingly,

$$ds^2 = g'_{\mu\nu}\, dx^\mu\, dx^\nu = 0 \tag{2}$$

with $x^0 = \tau v$. Since the universe expands radially (it is assumed to be homogeneous and isotropic), it is convenient to use spherical coordinates $x^k = (R, \theta, \phi)$ and thus $d\theta = d\phi = 0$. Equation (2) reduces to

$$\frac{dR}{dv} = \tau \sqrt{\frac{g'_{00}}{g'_{11}}} \tag{3}$$

This is Hubble's law taking into account gravitation, and hence dilation and curvature. When gravity is negligible, $g'_{00} \approx g'_{11} \approx 1$, thus $dR/dv = \tau$, and by integration, $R = \tau v$ or $v = H_0 R$ when the initial conditions are chosen appropriately.

4. CONSTRAINTS ON MOTION OF STARS

A star moving around the galaxy experiences the expansion of the universe. This is a constraint on the dynamical system that should be taken into account, and without which the theory is invalid.

The expansion of the universe causes an increase in the distance between the star and the center of the galaxy. But when this distance increases, the circular velocity changes accordingly. This constraint on the dynamical system should be taken into account along with the centrifugal formula $v_c^2 = GM/r$.

In this paper we derive the extra relation and show that it yields a term of the form $v_c^4 \propto M$.

5. EQUATIONS OF MOTION

In Einstein's general relativity theory the equations of motion follow from the vanishing of the covariant divergence of the energy-momentum tensor. This is a result of the restricted Bianchi identities. The equations obtained are usually geodesic equations. By means of a successive approximation in v/c, one obtains the Newtonian equation of motion and its generalization to a higher accuracy [10–27].

Accordingly, one has

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \tag{4}$$

We now find the lowest approximation of Eq. (4) in terms of t/τ, where t is a characteristic cosmic time and τ is the Hubble time, using the Einstein–Infeld–Hoffmann method [11, 12].

To this end we change variables in Eq. (4) from s to v, where v is related to the velocity-like coordinate x^0 by $x^0 = \tau v$. A simple calculation gives

$$\frac{d^2x^k}{dv^2} + \left(\Gamma^k_{\alpha\beta} - \frac{1}{\tau}\frac{dx^k}{dv}\Gamma^0_{\alpha\beta}\right)\frac{dx^\alpha}{dv}\frac{dx^\beta}{dv} = 0 \tag{5}$$

with $k = 1, 2, 3$. One can neglect the second term in the parentheses since it is one order smaller than the first, and thus

$$\frac{d^2x^k}{dv^2} + \Gamma^k_{\alpha\beta}\frac{dx^\alpha}{dv}\frac{dx^\beta}{dv} \approx 0 \tag{6}$$

The second term is equal to

$$\Gamma^k_{00}\left(\frac{dx^0}{dv}\right)^2 + 2\Gamma^k_{0b}\frac{dx^0}{dv}\frac{dx^b}{dv} + \Gamma^k_{ab}\frac{dx^a}{dv}\frac{dx^b}{dv} \tag{7}$$

But $x^0 = \tau v$, thus the second and third terms may be neglected with respect to the first, and we obtain

$$\frac{d^2x^k}{dv^2} + \tau^2\Gamma^k_{00} \approx 0 \tag{8}$$

The Christoffel symbol can be calculated also,

$$\Gamma^k_{00} = \frac{1}{2}g^{k\rho}(2\partial_0 g_{\rho 0} - \partial_\rho g_{00}) \tag{9}$$

where primes are omitted for brevity. Again we have a x^0-derivative $\partial_0 = \tau^{-1}\partial_v$ which is of higher order in t/τ, thus

$$\Gamma^k_{00} \approx -\frac{1}{2}g^{k\rho}\partial_\rho g_{00} \approx -\frac{1}{2}g^{ks}\partial_s g_{00} \tag{10}$$

Since $g^{ks} \approx \eta^{ks} = -\delta^{ks}$, we obtain

$$\Gamma^k_{00} \approx \frac{1}{2}\frac{\partial g_{00}}{\partial x^k} \tag{11}$$

and thus the geodesic equation yields

$$\frac{d^2x^k}{dv^2} + \frac{\tau^2}{2}\frac{\partial g_{00}}{\partial x^k} \approx 0 \tag{12}$$

Writing now $g_{00} = 1 + 2\phi/\tau^2$, we then obtain

$$\frac{d^2x^k}{dv^2} = -\frac{\partial\phi}{\partial x^k} \tag{13}$$

for the equations of motion in the lowest approximation. It remains to find the function $\phi(x)$.

6. FIELD EQUATIONS

To find the function ϕ, we have to solve the gravitational field equations. The question arises: What field equations does the metric tensor $g'_{\mu\nu}$ satisfy? We *postulate* that $g'_{\mu\nu}$ satisfies the Einstein field equations

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}T\right) \tag{14}$$

where $T = T_{\rho\sigma}g^{\rho\sigma}$ and

$$\kappa = \frac{8\pi k}{\tau^4} \tag{15}$$

with $k = G(\tau^2/c^2)$. We then have

$$T = T_{\mu\nu}g^{\mu\nu} \approx T_{\mu\nu}\eta^{\mu\nu} \approx T_{00}\eta^{00} = T_{00} \tag{16}$$

Thus we obtain

$$R_{00} = \kappa\left(T_{00} - \frac{1}{2}g_{00}T\right) \approx \frac{1}{2}\,\kappa T_{00} = \frac{1}{2}\,\kappa\tau^2\rho(x) \tag{17}$$

where $\rho(x)$ is the mass density.

The approximate value of R_{00} is

$$R_{00} = \frac{\partial\Gamma^{\rho}_{00}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}_{0\rho}}{\partial x^{0}} + \Gamma^{\rho}_{00}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{0\rho}\Gamma^{\rho}_{0\sigma} \approx \frac{\partial\Gamma^{\rho}_{00}}{\partial x^{\rho}} \approx \frac{\partial\Gamma^{s}_{00}}{\partial x^{s}} \tag{18}$$

Using now Eq. (11) for the value of the Christoffel symbol, we obtain

$$R_{00} \approx \frac{\partial\Gamma^{s}_{00}}{\partial x^{s}} = \frac{1}{2}\,\frac{\partial^2 g_{00}}{\partial x^s\partial x^s} = \frac{1}{2}\,\nabla^2 g_{00} \approx \frac{1}{\tau^2}\,\nabla^2\phi \tag{19}$$

where ∇^2 is the ordinary Laplace operator.

Equations (17) and (19) then give

$$\nabla^2\phi = \frac{1}{2}\,\kappa\tau^4\rho \tag{20}$$

or, using Eq. (15),

$$\nabla^2\phi(x) = 4\pi k\rho(x) \tag{21}$$

This equation is exactly the Newtonian equation for gravity but with $k = G\tau^2/c^2$ replacing the Newtonian constant G.

7. INTEGRATION OF THE EQUATIONS OF MOTION

The integration of the equation of motion (13) is identical to that familiar in classical Newtonian mechanics. But there is an essential difference which should be emphasized.

In Newtonian equations of motion one deals with a path of motion in the 3-space. In our theory we do not have that situation. Rather, the paths here indicate locations of particles in the sense of the Hubble distribution, which now takes a different physical meaning. With that in mind we proceed as follows.

Equation (13) yields the first integral

$$\left(\frac{ds}{dv}\right)^2 = \frac{kM}{r} \tag{22}$$

where v is the velocity of the particles, in analogy to the Newtonian

$$\left(\frac{ds}{dt}\right)^2 = \frac{GM}{r} \tag{23}$$

In these equations s is the length parameter along the path of accumulation of the particles.

Comparing Eqs. (22) and (23), and remembering that $k = G\tau^2/c^2$, we obtain

$$\frac{ds}{dv} = \frac{\tau}{c}\frac{ds}{dt} \tag{24}$$

Thus

$$\frac{dv}{dt} = \frac{c}{\tau} \tag{25}$$

Accordingly, we see that the particle experiences an acceleration $a_0 = c/\tau = cH_0$ directed outward when the motion is circular.

8. EFFECTIVE POTENTIAL

The motion of a particle in a central field is best described in terms of an “effective potential” V_{eff}. In Newtonian mechanics this is given by [28]

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{L^2}{2r^2} \tag{26}$$

where L is the angular momentum per mass unit. In our case the effective potential is

$$V_{\text{eff}}\,(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} + a_0 r \tag{27}$$

The circular motion is obtained at the minimal value of (27), i.e.,

$$\frac{dV_{\text{eff}}}{dr} = \frac{GM}{r^2} - \frac{L^2}{r^3} + a_0 = 0 \tag{28}$$

with $L = v_c r$, and v_c is the rotational velocity. This gives

$$v_c^2 = \frac{GM}{r} + a_0 r \tag{29}$$

Thus

$$v_c^4 = \left(\frac{GM}{r}\right)^2 + 2GMa_0 + a_0^2 r^2 \tag{30}$$

where $a_0 = c/\tau = cH_0$.

9. CONCLUDING REMARKS

The first term on the right-hand side of Eq. (30) is purely Newtonian, and cannot be avoided by any reasonable theory. The second one is the Tully–Fisher term. The third term is extremely small at the range of distances of stars around a galaxy.

It has been shown by Milgrom [3] that a term of the form $GMa_0 = GMcH_0$ can explain most of the observations of the dynamics of stars around the galaxies. The "modified Newtonian law of motion" proposed by him was found by adding arbitrarily an attractive force term for very far distances.

In conclusion it appears that there is no necessity for the existence of halo dark matter around galaxies. Rather, the results can be described in terms of the properties of spacetime [29].

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